Störmer Theory Applied to Magnetic Spacecraft Shielding

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Abstract. The existence of a toroidal region from which charged particles are excluded was demonstrated by \textit{Störmer} [1955] for a dipole magnetic field geometry. Using a standard numerical code to trace the trajectories of particles in a magnetic field, we find excellent agreement with the predicted region for a variety of particle energies and masses. The ability of magnetic fields to shield certain regions from energetic particles, such as galactic cosmic rays (GCRs), has led to the suggestion that certain magnetic field configurations could be used to shield the occupants of a spacecraft from the harmful effects of GCRs during interplanetary space travel. In particular, systems involving a deployed superconducting coil of wire that extends well beyond the dimensions of the spacecraft have been proposed as a viable solution to shielding GCRs. The correct use of the analysis by \textit{Störmer} [1955] requires that the radius of such a coil be much smaller than the dimension of the region being shielded. Alternatively, it is shown that the energy of the particles shielded from a given region decreases as the radius of a coil with a constant magnetic moment increases. The reality is that large magnetic fields, and thus currents, are necessary to adequately shield GCRs – neither of which are provided by deployed magnetic shields.

1. Introduction

Strategies for protecting spacecraft inhabitants from the harmful effects of GCRs include both passive and active techniques. Passive techniques essentially entail using solid material to create a shield which prevents particles from penetrating a given region by absorbing the energy of incident particles. Such a shield is all but impractical due to the mass and energy required for a shield sufficiently massive to protect from GCRs with energies >1 GeV [e.g., Spillantini, 2000].

Active shielding eliminates the need for massive shields by using electric or magnetic fields to deflect particles from a region surrounding the spacecraft. However, a different set of issues both technical and practical are presented by these shielding strategies. The most serious is the safety concern due to the exceptionally large voltages (>1 kV) or large magnetic fields (>1 T) in close proximity to the spacecraft occupants that are required to shield the hazardous GCRs [e.g., Townsend, 2000]. The potential dangers of such strategies could exceed the harm caused by the energetic particles for which the shield is intended.

Advances in high-temperature superconductors have led some to advocate the promise of a deployed coil of wire that extends far beyond the confines of the spacecraft to provide a magnetic shield. Proponents of such a solution point to the reduced current, energy, and mass required for such a system to maintain a given magnetic dipole moment over traditional systems confined to the spacecraft [Cocks, 1991; Cocks et al., 1997]. Unfortunately the analysis used in these studies to justify the shielding capacity of a deployed superconducting coil is in error [c.f., Shepherd and Kress, 2007]. The authors of these studies mistakenly assume that the magnetic field of a loop of wire is equivalent to a pure dipole field, which is only true for distances much larger than the radius of the current loop.

It has been shown by \textit{Störmer} [1955] (hereafter referred to simply as Störmer) that a toroidal region exists around the center of a dipole magnetic field from which particles of a given energy are excluded. We use a numerical particle tracing code to demonstrate the existence of this region and show that the shielded region near the center of current loop is diminished as the radius of the coil increases. The decrease in shielding capacity of a deployed coil can be interpreted as either an increasingly poor approximation between the field of a current loop and a dipole resulting in a reduced ability to shield particles of a given energy near the center of the loop, or that the energy of the particles that can be shielded by such a configuration decreases. Our analysis suggests that deployed current carrying coils are ineffective as magnetic shields from GCRs.

We first demonstrate the excellent agreement between our numerical results and the analysis performed by \textit{Störmer} for an ideal dipole magnetic field (hereafter referred to as Störmer theory); section 2 provides a brief description of the important results from Störmer theory; section 3 describes the numerical technique used to trace particles as well as the results of the numerical simulation for a dipole magnetic field. In section 4 the numerical technique is applied to the magnetic fields produced by deployed coils of various radii and the shielding capabilities of these devices is discussed in section 6.

2. \textit{Störmer} Magnetic Shielding

For certain magnetic field configurations the influence these fields impart on charged particles via the Lorentz force can lead to regions of space for which particles below a certain energy are forbidden access. These regions of space are said to be shielded from such particles. Using the concept of a magnetic potential barrier, \textit{Störmer} showed the existence of a shielded region for a dipole magnetic field configuration.

The equation of motion of a particle with charge \(q\) and moving with velocity \(\mathbf{v}\) in a static magnetic field \(\mathbf{B}(\mathbf{r})\) and no electric field is given by

\[
\frac{d\mathbf{v}}{dt} = \frac{q}{\gamma m} \mathbf{v} \times \mathbf{B}(\mathbf{r}),
\]

where \(m\) is the rest mass of the particle, \(\gamma = (1 - v^2/c^2)^{-1/2}\) is the relativistic correction factor, and \(c\) is the speed of light; SI units are used throughout. While \(\gamma\) is more generally part of the derivative in equation (1), the Lorentz force in the case of a static magnetic field with no electric field is always perpendicular to the particle trajectory and the speed of the particle is therefore constant.

In general, no closed form solution of equation (1) has been found except for a few special cases. \textit{Störmer} showed the existence of a

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magnetic potential barrier in a dipole magnetic field, which shields charged particles entering from a large distance away. Stormer begins his mathematical development with equation (1), first finding two integrals of the motion; the energy and the azimuthal component of the generalized momentum. These dynamical constants are then shown to bound the particle motion to certain regions of space.

The azimuthal component of the generalized momentum of the particle is given by

\[ p_\phi = m \rho \dot{\phi} + \frac{q}{c} \rho A_\phi \]  

(2)

with cylindrical coordinates \((\rho, \phi, z)\) used and \(A = A_\phi \hat{\phi}\) is the magnetic vector potential. By inserting the analytical expression for the vector potential of a pure dipole \(A_\phi = -\frac{M\rho}{r^3}\), with dipole moment \(M = M_0 \hat{z}\) into equation (2), Stormer shows the existence of an inner forbidden region that a particle arriving from a large distance away cannot access. The boundary of this region is defined by the equation

\[ r = \frac{M \mu_0}{4\pi \gamma m v} \frac{\cos^2 \lambda}{1 + \cos^2 \lambda} \]  

(3)

where \(\lambda\) is the magnetic latitude \((\pi/2 - \theta)\) and \(r\) is the radial distance from the center of the dipole in spherical coordinates.

From equation (3), we see that the extent of the surface bounding the forbidden region for a given \(M\) is determined by the so-called rigidity of the particle \(R \equiv \gamma m v/c\). The factor

\[ C_{st} \equiv \sqrt{\frac{M \mu_0}{4\pi \gamma m v}} = \sqrt{\frac{M \mu_0}{R}} \frac{c}{4\pi} \]  

(4)

has the dimensions of length and is called the Störmer length. This quantity is sometimes used as a length-scale to characterize the size of the shielded region. Note, however, that the shielded region is somewhat smaller; the toroidal region defined by equation (3) has an outer radius of approximately 0.4 \(C_{st}\) for \(\lambda = 0\).

3. Numerical Technique

In order to numerically solve equation (1) for a given static magnetic field configuration, we use a standard Runge-Kutta 4th order method to solve the coupled set of six first-order ODEs for the Cartesian position \((r = x\hat{i} + y\hat{j} + z\hat{k})\) and velocity \((v = v_x \hat{i} + v_y \hat{j} + v_z \hat{k})\) of a single particle. An adaptive time step is used by adjusting the step size to be 0.1% of the instantaneous gyroperiod of the particle. A maximum step size of 1 \(\mu\)s is used, but no lower bound on the minimum step size is imposed. In the case of a particle approaching the center of a dipole located at the origin, where the field becomes unbounded, the step size is allowed to become as small as necessary to resolve the trajectory. Such trajectories can take exceptional long to compute, but are nevertheless allowed to run to completion and no trajectories are discarded.

Two basic configurations of the static magnetic field are used in our numerical simulations. In the case of a dipole, the magnetic field is determined at each time step from the exact analytic expression for a dipole field. The magnetic field due to a circular loop of wire, on the other hand, must be approximated at each time step. We use the Biot-Savart Law to compute the magnetic field at the given location of the particle that is due to a straight-line segment of the loop. The total field is obtained by summing the contributions from each segment of the loop. For the examples shown in this study, a segment of 1° was chosen (which corresponds to a segment length of \(\sim 17\) m for a 1-km radius loop). Tests were performed at a higher angular resolution with no observable differences noted in the results. All of the calculations used in this study were performed on a standard desktop personal computer.

In order to determine any regions which are shielded due to the field of a magnetic dipole, the trajectories of a collection of particles of mass \(m\) and energy \(KE\) are determined using the numerical technique described. To sample the possible trajectories which could penetrate the region near the origin of a magnetic dipole, particles are initiated with a speed given by their mass \(m\) and energy \(KE\) and located at random positions uniformly distributed on the surface of a sphere of radius \(R_0\). The initial velocity of the particles is directed toward a circular target located at the origin. To sample the appropriate trajectories which could possibly penetrate the shielded region, the direction of the velocity vector is determined by selecting another uniformly distributed random point on the target of radius \(\delta r\). The initial velocity vectors are thus sampled from a cone around the initial position vector. The selection of particle trajectories is not intended to be representative of a true distribution function, but rather an appropriate sampling of points necessary to illuminate any shielded regions located near the origin.

For the simulations used in this study \(R_0\) was chosen to be 50 km. The magnetic field strength at a distance of \(R_0\) in the plane of a dipole with \(M = 1.1 \times 10^{-5}\) A m\(^2\) (a value which is used in many of the examples used in this study) is \(\sim 9\) nT, comparable to the strength of a moderate interplanetary magnetic field (IMF). The
value of $\delta r$ must be chosen to be large enough such that suitable trajectories are sampled to sufficiently determine the bounded region due to a given magnetic field. This value depends on the particle energy, charge, and magnetic field configuration. Selecting a value of $\delta r$ that is too large merely adds trajectories that pass far from the shielded region. For many of the examples used in this study a value of $\delta r = 500$ m was deemed appropriate.

Figure 1 shows the computed trajectories of 100 sample particles near the origin of a magnetic dipole. The color of the particle paths indicates the radial distance from the origin; red indicates the closest approach of a particle. The shielded region due to a dipole magnetic field can be seen in Figure 1 to be roughly described by equation (3).

In order to obtain a more quantitative assessment of the agreement between the shielded region predicted by Störmer for a dipole magnetic field, and that described by the numerical results, a large number of particle trajectories were computed and the positions of the closest approach to the origin were determined. For this example, 1 GeV, Fe$^+$ particles with corresponding rigidity $R = 10.3$ GV in the field of an ideal magnetic dipole given by

$$B(r) = \frac{\mu_0}{4\pi} \left( \frac{M}{r^3} + \frac{3(M \cdot r)r}{r^5} \right)$$

with a magnetic dipole moment $M = 1.1 \times 10^{13} \hat{z} \text{ A m}^2$ were chosen. The positions $(\rho, z)$ of the closest approach to the origin of 50,000 trajectories are shown in Figure 2. The shielded region given by equation (3) and the Störmer length given by equation (4) are shown as solid and dashed lines in Figure 2, respectively for the specified values of $M$ and $R$. For this example, $r(\lambda = 0) \simeq 74$ m and $C_{st} \simeq 179$ m. The closeness of various trajectories to the predicted boundary and lack of trajectories penetrating this toroidal region clearly indicate an excellent agreement between the numerical results and Störmer theory.

The particular choice of values for $q, \gamma, m,$ and $v$ used in equation (3) are representative of GCR particles which present a health risk from radiation exposure to humans in interplanetary space [Cougnet et al., 2005]. Because the spectrum of GCRs is a continuum we have performed numerical simulations for both iron ions (Fe$^+$) and protons (H$^+$) for the energies 1 GeV, 10 GeV, and 100 GeV, which have the corresponding values of $\gamma$: 1.02, 1.19, and 2.91 for Fe$^+$ and 2.07, 11.72, and 108.2 for H$^+$, respectively. Figure 3 shows the excellent agreement between the shielded region obtained from our numerical simulation and the theory given by equation (3) over a range of particle energies and values of $\gamma$. 

Figure 3. Positions of the closest approach to the origin for 50,000 particles showing good agreement with the toroidal Störmer region for Fe$^+$ ions with energy (a) 1 GeV, (b) 10 GeV, and (c) 100 GeV; as well as protons (H$^+$) with energy (d) 1 GeV, (e) 10 GeV, and (f) 100 GeV.
5. Discussion

Two things are immediately obvious from Figure 5. The first is that the deployed coil does indeed provide a shielded region from particles of this rigidity. The region, however, is clearly located near the wire rather than in the center of the coil where it is desired. This fact makes perfect sense since the magnetic field is strongest near the wire and thus deflects the particles more effectively. In fact, as the radius of the coil increases the shielded region near the wire becomes a circular region equivalent to that of an infinitely long straight wire carrying current I. Because I is proportional to a−2 for a constant magnetic dipole moment M the size of the shielded region decreases as the size of the coil increases.

The second important thing to note about the numerical results shown in Figure 5 is that the shielded region determined by Störmer theory for a given particle rigidity is better approximated as the radius a of the coil decreases. Specifically, the numerical results suggest that the analysis holds only when the condition

\[ a \ll C_{st} \]

is satisfied. If equation (7) holds then the magnetic field that a particle experiences along its entire trajectory can be approximated by the field due to a magnetic dipole and Störmer-like shielding can, therefore, be expected.

Strictly speaking, for Störmer theory to be valid for all particle trajectories, the radius of the coil must approach zero. It is evident from Figure 5 that the approximation to the Störmer region described by equation (3) is poorest for trajectories approaching the origin along the polar axis (\( \lambda = \pm \pi/2 \)). For example, Figure 5e shows that a Störmer-like region exists when \( a \approx 0.1 C_{st} \), however, a few particles with trajectory approaches near the poles (\( \lambda \approx \pm \pi/2 \)) penetrate the Störmer boundary nearest the origin. The region is, therefore, not entirely shielded from particles of the given energy, particularly from those with trajectories nearly along the axis of the dipole moment M.

Assuming that equation (7) is satisfied, it is possible to demonstrate that for a constant magnetic dipole moment M the energy of the particles that can be shielded from a Störmer-like region decreases as the radius of the coil is increased. Considering first particles with \( \gamma \gg 1 \), it is easy to show from equations (4) and (7) that

\[ KE \simeq \gamma mc^2 \sim \frac{1}{a^2} \]
That is, the energy of the particles that are shielded from the region given by equation (3) are inversely proportional to the square of the radius of the coil. The reduction factor is even greater for lower energy particles with $\gamma \approx 1$. In this case it is possible to show that

$$KE \simeq \frac{1}{2}mv^2 \sim \frac{1}{a^4}$$

So while the energy and mass requirements of a deployed coil decrease with increasing radius, as stated by Cocks et al. [1997], so too does the energy of the particles that are shielded from the desired region.

It is also worth noting that for a given $M$, the magnitude of the magnetic field at the center of the coil is proportional to $a^{-3}$. The result is that as the radius of the coil is decreased, the approximation to Störmer theory becomes better, but the strength of the magnetic field near the center of the coil increases dramatically. Table 1 shows the strengths of the magnetic field at the centers of coils of various radius for the magnetic moment used in the other simulations ($M = 1.1 \times 10^{13}$ A m$^2$). For this scenario $C_{st} \approx 180$ m. The last two rows in Table 1 are therefore the only coil radii for which equation (7) is satisfied and Störmer theory applies. The magnetic field strength near the center of such coils is quite large ($>10^3$ T). Even in the shielded region away from the coil the field is extremely high for these configurations. For example, the magnetic field strength at a distance $0.5r$ ($\lambda = 0$) (a position located near the center of the shielded toroidal region) for the 5-m radius coil in Table 1 is $>20$ T.

While there presently appear to be no existing standards on the levels of static magnetic fields that are deemed to pose no serious health risk, the World Health Organization (WHO) recently published an environmental health criteria on static fields (electric and magnetic) [EHC, 2006]. Although it is stated that the physiological effects of extended exposure to magnetic fields $>1$ T are not well understood, several studies cited in the monograph report volunteers experiencing vertigo and nausea while moving in static magnetic fields $>2$ T. In a typical magnetic resonance imaging (MRI) device, where patients remain motionless, magnetic fields are in the range 1–10 T. It is, therefore, entirely reasonable to assume that the strong magnetic fields ($>10^3$ T) present near the center of the 5-m and 10-m radius coils would most likely prove fatal to any nearby occupants.
6. Summary

Using a standard numerical technique we have computed particle trajectories in a variety of magnetic field configurations. The numerical results show excellent agreement with Störmer theory. For an ideal dipole magnetic field a shielded toroidal region exists for which particles of a given rigidity are excluded. This region, given by equation (3), is verified by tracing many particles of a given rigidity with various initial positions and velocities. Simulations were performed for a variety of different particle rigidities in a dipole magnetic field.

With the renewed interest in manned space exploration the problem of mitigating the harmful effects of GCRs during such space flights has also received increased attention. Many novel ideas in active shielding by electric or magnetic fields have been proposed [Sussingham et al., 1999]. Störmer theory has been used incorrectly in at least one case in order to suggest that a deployed coil of superconducting wire could be used to shield a spacecraft from GCRs [Cocks, 1991; Cocks et al., 1997]. The authors argue that coils of increasingly larger radius (up to 10 km) require less current, less energy, and less mass than more conventional coils confined to the actual spacecraft and are, therefore, a solution to shielding astronauts from the harmful effects of GCRs during interplanetary travel.

However, in their analysis Cocks et al. [1997] mistakenly equate the magnetic field of a current loop with that of an ideal magnetic dipole. The approximation is only valid for distances that are much larger than the radius of the current loop. Attempting to shield a region that is smaller than the radius of the current loop clearly violates this condition. We have shown numerically that no shielding occurs in the desired region for particles of a given rigidity when the radius of the loop is greater than that of the region defined by equation (3). A shielded region does exist for this situation, however, it is located near the wire where the magnetic field is strongest and deflection of particles is most effective. As the radius of the coil increases the shielded region better approximates that of an infinitely long straight wire, which becomes smaller as the current necessary to sustain a given magnetic moment decreases.

We assert, and demonstrate numerically, that in order to approximately achieve a shielded Störmer region, given by equation (3), using a loop of wire the radius of the loop must be much smaller than the radial dimension of the region being shielded. If this condition, equation (7), is satisfied the field of the loop may be accurately approximated by a dipole magnetic field at all points along the particle trajectory. Using equation (7) it is demonstrated that the energy of the particles shielded from such a Störmer-like region are proportional to $a^{-2}$ and $a^{-4}$ for particles with $\gamma \gg 1$ and $\gamma \approx 1$, respectively. So while the reduced current, energy, and mass requirements are correctly stated by Cocks et al. [1997] for coils of increasingly larger radius, one must also add the caveat that the energy of the particles shielded from a given region also decreases.

Based on our numerical simulations we conclude that the benefits of deployed magnetic shielding using a coil of wire are not warranted. GCRs are most effectively shielded from desired regions by strong magnetic fields where the deflecting force is correspondingly large, not by the weak magnetic fields of extended current loops. Our analysis leads us to join others [e.g., Parker, 2005, 2006] in doubting the practicality and feasibility of active magnetic shields in protecting astronauts from the harmful effects of GCRs during interplanetary travel.

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